Journal of Engineering Mathematics, Vol. 5, No. 1, January 1971
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

Gaussian Quadrature Formulas for the Numerical Integration of Bromwich's Integral and the Inversion of the Laplace Transform

R. PIESSENS

Institute for Applied Mathematics, University of Leuven, Heverlee, Belgium

(Received August 6, 1969 and in revised form May 15, 1970)

SUMMARY
An approximate formula for the inversion of the Laplace transform $F(p)$ is studied. The formula is exact whenever $F(p)$ is a linear combination of p^{-s+k}, $k=0, 1, 2, \ldots, 2N-1$, with s an arbitrary positive real number. The formula is derived from a gaussian integration formula for Bromwich's inversion integral.

A numerical example is given as illustration of the use of the approximate inversion formula.

1. Introduction

The Laplace transform is useful in solving some ordinary and partial differential equations and integral equations and arises in many areas of engineering mathematics. However, the exact determination of the original function $f(t)$ from its Laplace transform

$$F(p) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

is often a great difficulty. In many cases, numerical methods must be used. A number of interesting numerical methods are known [1], [2], [3], [4], [5], [6], the majority of which, unfortunately gives unsatisfactory results if $F(p)$ has a branch point at infinity.

In this paper, we shall consider an approximate formula for the inversion of the Laplace transform, namely,

$$f(t) = t^{s-1} \sum_{k=1}^{N} \{A_k (u_k/t)^s F(u_k/t)\} \tag{2}$$

where s is a positive real number that must be chosen so that $p^s F(p)$ is analytic and has no branch point at infinity, thus so that we can write

$$p^s F(p) = \sum_{k=0}^{\infty} a_k p^{-k} .$$

We shall give formulas for the coefficients A_k and we shall discuss a method for the calculation of the coefficients u_k. The coefficients A_k and u_k are dependent on the value of s and N. Salzer [6], [7] has already studied a formula of the same form of (2), but only for the special value $s=1$. Krylov and Skoblya [8] and Skoblya [9] have given tables for the coefficients A_k and u_k for several values of N and s, but with a very limited accuracy. Moreover, they have not given the formulas for A_k and u_k, presented here.

2. Inversion of the Laplace Transform by Integration of Bromwich's Integral

The original function $f(t)$ is given by

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{pt} F(p)\, dp \tag{3}$$

where c is chosen so that the line $\mathrm{Re}\,(p)=c$ lies to the right of all singularities of $F(p)$, but is otherwise arbitrary.

Substituting

$$pt = u$$

and

$$F(u/t) = u^{-s} G(u)$$

where s is the parameter defined in section 1, (3) yields

$$t f(t) = \frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} e^u u^{-s} G(u) du \tag{4}$$

where $c' = tc$.

Thus for the inversion of the Laplace transform, we need a formula to calculate the integral in the second member of (3). We consider the approximate formula

$$\frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} e^u u^{-s} G(u) du \simeq \sum_{k=1}^{N} A_k G(u_k) . \tag{5}$$

Indeed, substituting (5) in (4), we have the desired approximate formula (2) for the inversion of the Laplace transform. We just have to give convenient formulas for the coefficients A_k and u_k.

We try to determine the abscissas u_k and the weights A_k in (5) so that the integration is exact whenever $G(u)$ is a polynomial in u^{-1}, of degree $\leq 2N-1$. Then we have a Gaussian integration formula, that has a degree of precision $2N-1$. In section 3, we shall study orthogonal polynomials, the roots of which are the coefficients u_k. In section 6, we shall give formulas for the coefficientsA_k.

3. Orthogonal Polynomials Connected with the Gaussian Integration Formulas

From the theory of Gaussian quadrature formulae (Davis and Rabinowitz [10]), it is well known that the existence of a polynomial $P_{N,s}(p^{-1})$ in the variable p^{-1}, of degree N, with the property

$$\frac{1}{2\pi j} \int_L e^p p^{-s} P_{N,s}(p^{-1}) p^{-r} dp = 0 \tag{6}$$

for $r = 0, 1, 2, \ldots, N-1$, where L is an arbitrary vertical line in the positive half plane of the complex plane, is necessary and sufficient for the existence of an N-point integration formula with degree of precision $2N-1$. Condition (6) means that $P_{N,s}(p^{-1})$ is a polynomial of degree N, which is orthogonal to all polynomials of degree $< N$, with respect to the weight function $e^p p^{-s}$.

If we set

$$\Phi_r(t) = \mathscr{L}^{-1}\{p^{-(s+r)} P_{N,s}(p^{-1})\} \tag{7}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform, the condition (6) is equivalent to

$$\Phi_r(1) = 0$$

for $r = 0, 1, 2, \ldots, N-1$.

Using the convolution property of the Laplace transform, we obtain

$$c_N + \int_0^1 u^{s+r-1} q_{N-1,s}(1-u) du = 0 , \qquad r = 0, 1, \ldots, N-1 \tag{8}$$

where c_N and $q_{N-1,s}(t)$ are defined by the relations

$$P_{N,s}(p^{-1}) = A[c_N + p^{-1} Q_{N-1,s}(p^{-1})] \tag{9}$$

and

$$q_{N-1,s}(t) = \mathscr{L}^{-1}\{p^{-1} Q_{N-1,s}(p^{-1})\}$$

with $Q_{N-1,s}$ a polynomial in p^{-1} of degree $N-1$ and A a normalization coefficient which will be determined later.

Relation (8) implies

$$\int_0^1 u^{s+r-1}(1-u)q_{N-1,s}(1-u)du = 0\,, \qquad r=0, 1, \ldots, N-2\,.$$

Thus, $q_{N-1,s}(1-u)$ is orthogonal on [0, 1] to all polynomials of degree $< N-1$, with respect to the weight function $w(u)=u^{s-1}(1-u)$.

Hence, we have

$$q_{N-1,s}(t) = P_{N-1}^{(1,s-1)}(1-2t) \tag{10}$$

where $P_N^{(\alpha,\beta)}(x)$ denotes a Jacobi polynomial of degree N (see [11], section 10.8).

Consequently, we have

$$q_{N-1,s}(t) = {}_2F_1(-N+1, N+s; 2; t)$$

and

$$c_N = -\int_0^1 u^{s-1}\,{}_2F_1(-N+1, N+s; 2; 1-u)du$$

or

$$c_N = -s^{-1}\,{}_3F_2\left[\begin{matrix}-N+1, & N+s, 1 \\ 2 & , s+1\end{matrix}\; ; 1\right] = -\frac{1}{N(N+s-1)}\,. \tag{11}$$

Substituting (10) and (11) in (9), we have

$$P_{N,s}(p^{-1}) = -\frac{A}{N(N+s-1)} + \frac{A}{p}\,{}_3F_1\left[\begin{matrix}-N+1, & N+s, & 1 \\ 2 & & \end{matrix}\; ; \frac{1}{p}\right]. \tag{12}$$

If we choose the standardization factor as

$$A = (-1)^{N+1}N(N+s-1)$$

we obtain finally

$$P_{N,s}(p^{-1}) = (-1)^N\,{}_2F_0(-N, N+s-1; -; p^{-1})\,. \tag{13}$$

Formula (13) is valid for all real $s>0$.

For $s>1$ however, the derivation can be simplified and it is easy to demonstrate that

$$P_{N,s}(p^{-1}) = p\,\mathscr{L}\{P_N^{(0,s-2)}(1-2t)\} \qquad (s>1)\,. \tag{14}$$

Another interesting form of $P_{N,s}(p^{-1})$ is the confluent hypergeometric function

$$P_{N,s}(p^{-1}) = p^{-N}(N+s-1)_N\,{}_1F_1(-N;\ 2-2N-s;\ -p) \tag{15}$$

where

$$(q)_N = \frac{\Gamma(q+N)}{\Gamma(q)}\,. \qquad \text{(Pochhammer's symbol)}$$

4. Properties of the Orthogonal Polynomials

4.1. *Differential Equation.*

It is well known [11] that the generalized hypergeometric function

$$u = {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_p \\ \beta_1, \beta_2, \beta_3, \ldots, \beta_q\end{matrix}\; ; x\right] \tag{16}$$

satisfies the differential equation

$$\{\delta(\delta+\beta_1-1)\dots(\delta+\beta_q-1)-x(\delta+\alpha_1)\dots(\delta+\alpha_p)\}\,u=0 \tag{17}$$

where

$$\delta = x\,\frac{d}{dx}\,.$$

Thus, $u=P_{N,s}(x)$ is a solution of

$$x^2u''+(sx-1)u' = N(N+s-1)u\,. \tag{18}$$

The differential equation (18) is a special case of

$$x^2u''+(ax+b)u' = N(N+a-1)u \tag{19}$$

which is studied by Krall and Frink [12]. Solutions of (19) are the generalized Bessel polynomials $Y_N(x, a, b)$. Thus, the polynomials $P_{N,s}(x)$ are special cases of the generalized Bessel polynomials

$$P_{N,s}(x)=(-1)^N\,Y_N(x,\,s,\,-1)\,. \tag{20}$$

The properties 4.2, 4.3 and 4.4 are easily derived from the corresponding properties of the generalized Bessel polynomials. Properties 4.3 and 4.4 can also be obtained from known properties of confluent hypergeometric functions.

4.2. *Rodrigues' Formula*

$$P_{N,s}(p^{-1})=(-1)^N e^{-p}p^{N+s-1}\,\frac{d^N}{dp^N}\,(e^p\,p^{-N-s+1}) \tag{21}$$

4.3. *Recurrence Relation*

The most important recurrence relation for the polynomials $P_{N,s}(x)$ is

$$P_{N,s}(x)=(a_{N,s}x+b_{N,s})\,P_{N-1,s}(x)+c_{N,s}P_{N-2,s}(x) \quad \text{for} \quad N\geqq 2 \tag{22}$$

where

$$a_{N,s}=\frac{(2N+s-3)(2N+s-2)}{N+s-2} \quad b_{N,s}=\frac{(2N+s-3)(2-s)}{(N+s-2)(2N+s-4)} \quad c_{N,s}=\frac{(2N+s-2)(N-1)}{(N+s-2)(2N+s-4)}$$

and

$$P_{0,s}(x)=1\,, \quad P_{1,s}(x)=sx-1\,.$$

4.4. *Expression for the Derivative*

$$\frac{d}{dp}\,P_{N,s}(p^{-1})=-\left(Np^{-1}+\frac{N}{2N+s-2}\right)P_{N,s}(p^{-1})-\frac{N}{2N+s-2}\,P_{N-1,s}(p^{-1})\,. \tag{23}$$

4.5. *Inverse Laplace Transform*

Using (13), we obtain

$$\mathscr{L}^{-1}\{p^{-a}P_{N,s}(p^{-1})\}=(-1)^N\,\frac{t^{a-1}}{\Gamma(a)}\,{}_2F_1(-N,\,N+s-1\,;\,a\,;\,t) \tag{24}$$

where a is an arbitrary real positive number.

4.6. *Moments*

We define the r-th moment as

$$M_{N,r}=\frac{1}{2\pi j}\int_L e^p p^{-s}P_{N,s}(p^{-1})p^{-r}dp\,. \tag{25}$$

It is obvious that

$$M_{N,r}=0 \quad \text{for} \quad r=0,1,2,\ldots,N-1\,.$$

Further, we have

$$M_{N,N+k}=\frac{(-1)^N(N+k)!}{\Gamma(2N+k+s)k!} \quad \text{for} \quad k=0,1,2,\ldots \tag{26}$$

To demonstrate (26), we remark that

$$M_{N,r}=\Phi_r(1)$$

where $\Phi_r(t)$ is defined as in (7).

Setting $t=1$ in (24), we obtain

$$M_{N,r}=\frac{(-1)^N}{\Gamma(r+s)}\,{}_2F_1(-N,N+s-1;\ r+s;\ 1)$$

or

$$M_{N,r}=\frac{(-1)^N}{\Gamma(r+s)}\,\frac{\Gamma(r+s)\Gamma(r+1)}{\Gamma(r+s+N)\Gamma(r+1-N)}\,.$$

Thus, formula (25) is proved.

Corrollary

The coefficient of x^N in $P_{N,s}(x)$ is

$$\frac{\Gamma(2N+s-1)}{\Gamma(N+s-1)}\,. \tag{27}$$

Using (26) with $k=0$, and (27), we obtain the important expression

$$h_{N,s}=\frac{1}{2\pi j}\int_L e^p p^{-s}[P_{N,s}(p^{-1})]^2dp=\frac{(-1)^N N!}{(2N+s-1)\Gamma(N+s-1)}\,. \tag{28}$$

5. Abscissas of the Gaussian Integration Formulas

The abscissas of the Gaussian integration formula of order N are the zeros of $P_{N,s}(p^{-1})$. Grosswals [13] has studied certain properties concerning the zeros of the simple Bessel polynomials. Some of these properties are also valid for the generalized Bessel polynomials, [14]. The quadrature formulas of even order have no real abscissas, those of odd order have exactly one real abscissa. All the abscissas have positive real parts and occur in complex conjugated pairs.

The zeros of $P_N(u^{-1})$ must be found by a numerical process of solving a polynomial for its roots, for instance Newton–Raphson's iteration process. Finding an approximate zero as starting value for the iteration process is facilitated by a certain regularity in the distribution of the zeros. For fixed N and s, the zeros lie very nearly on a circle with center on the negative real axis. The radius of this circle is approximately an increasing linear function of N and s. See fig. 1 and fig. 2. For fixed N and s, the angular distance between two consecutive abscissas is nearly constant (see [15]).

6. Weights of the Gaussian Integration Formulas

We shall give two formulas for the weights. These formulas can be proved in the same way as for the classical Gaussian integration formulas.

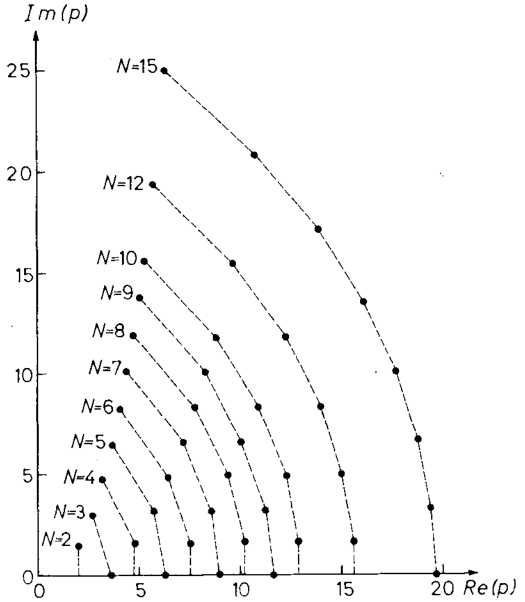

Figure 1. Distribution of the Gaussian abscissas in the first quadrant of the complex plane for $s=1$ and various values of N.

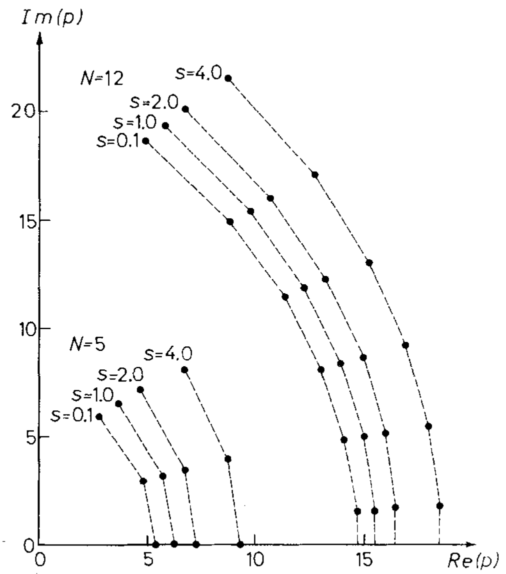

Figure 2. Distribution of the Gaussian abscissas in the first quadrant of the complex plane for $N=5$ and $N=12$ and various values of s.

(i) *First formula for the k-th weight* A_k

$$A_k = \left[\sum_{M=0}^{N-1} \frac{1}{h_{M,s}} [P_{M,s}(u_k^{-1})]^2 \right]^{-1} \tag{29}$$

where $h_{M,s}$ is given by (28) and where u_k is the k-th abscissa.

(ii) *Second formula for the k-th weight*

Using the relations (23) and (26) and the Christoffel–Darboux formula

$$\sum_{M=0}^{N} \frac{P_{M,s}(p^{-1}) P_{M,s}(y^{-1})}{h_{M,s}} = \frac{P_{N+1,s}(p^{-1}) P_{N,s}(y^{-1}) - P_{N+1,s}(y^{-1}) P_{N,s}(p^{-1})}{a_{N+1,s} h_{N,s}(p^{-1} - y^{-1})}$$

where $h_{N,s}$ is given by (28) and $a_{N+1,s}$ is the coefficient which arises in the recurrence relation

(22), formula (29) can be transformed in

$$A_k = (-1)^{N-1} \frac{(N-1)!}{\Gamma(N+s-1)Nu_k^2} \left[\frac{2N+s-2}{P_{N-1,s}(u_k^{-1})}\right]^2 . \tag{30}$$

7. Some Practical Remarks

1. No convenient rule exists for determining at the outset the order N so that the desired accuracy is obtained. The practical procedure is the use of a series of formulas (2) with increasing order. If, for the given value of t, agreement occurs of two successive approximations to within the desired accuracy, the last computed value is retained as definitive result. This procedure has the disadvantage of using different values of the arguments of the Laplace transform for different values of the order of formula (2). Increase in the order of the formula makes no use of previous evaluations of the Laplace transform $F(p)$. This disadvantage can be avoided if, in combination with the Gaussian rules, new quadrature formulas are used, given by Piessens [16].
2. When the order N of the formula (2) is large, the moduli of the coefficients A_k are also large. This may lead, from the numerical point of view, to large losses of significance by cancellation.
3. Several authors have calculated tables of the coefficients A_k and u_k. We give here a survey of these tables. Krylov and Skoblya give in [8] the coefficients A_k and u_k to 5 to 7 significant figures for $N=1(1)9$ and for values of s which are integer multiples of $\frac{1}{4}$ and $\frac{1}{3}$ between the limits $\frac{1}{4}$ and 3. In [19], they give for $s=1(1)5$ and $N=1(1)15$ the A_k's and u_k's to 20 significant figures and for $s=0.01(0.01)3.00$ and $N=1(1)10$ to 7 to 8 significant figures.

Skoblya [9] gives the u_k's to 8 significant figures and the A_k's to 7 significant figures for $N=1(1)10$ and $s=0.1(0.1)3.0$.

Salzer [6] gives the coefficients u_k, their inverses, and the products $u_k A_k$ to between 4 and 8 significant figures for $N=1(1)8$ and $s=1$.

In another paper, Salzer [7] gives the same quantities to between 12 and 15 significant figures for $N=1(1)16$ and $s=1$.

Stroud and Secrest [17] give the abscissas u_k and the weights A_k to 30 significant figures for $N=2(1)24$ and $s=1$.

Piessens [18] gives the u_k's and the A_k's to 16 significant figures for $N=6(1)12$ and $s=0.1$ (0.1)3.0, 3.5, 4.0, $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}$.

8. A Numerical Example

The first step in using formula (2) is the determination of the value of s. In most cases, there is no difficulty. However, for some Laplace transforms, there exists no convenient value of s. Sometimes, $F(p)$ can then be written as a sum of two or more Laplace transforms that can be inverted separately, each with a different value of s.

For instance

$$F(p) = p^{-0.5} \exp(-p^{-0.5}) \tag{31}$$

can be written as

$$F(p) = p^{-0.5} \cosh(p^{-0.5}) - p^{-0.5} \sinh(p^{-0.5}) . \tag{32}$$

Here, the first term of the second member can be inverted with $s=0.5$ and the second term with $s=1.0$.

The exact original function is

$$f(t) = \frac{1}{2t\sqrt{(\pi t)}} \int_0^\infty u \exp(-u^2/4t) J_0(2\sqrt{u}) du .$$

Using the approximate solution

$$f(t) = t^{-0.5} \sum_{k=1}^{6} A_k^{(0.5)} G_1(u_k^{(0.5)}) - \sum_{k=1}^{6} A_k^{(1.0)} G_2(u_k^{(1.0)}) \tag{33}$$

where

$$G_1(x) = \cosh [\sqrt{(t/x)}]$$

and

$$G_2(x) = \sqrt{(x/t)} \sinh [\sqrt{(t/x)}]$$

and where $A_k^{(\alpha)}$ and $u_k^{(\alpha)}$ are the weights and abscissas for $s=\alpha$ and $N=6$, given in [18], we obtain the following errors (by error we mean | true value − approximate value |)

t	exact original function	error
1.0	−0.01072342858155	$<1.10^{-14}$
10.0	−0.02478598394520	1.10^{-14}
20.0	−0.00308187970796	5.10^{-14}
50.0	0.00271995009362	7.10^{-14}
100.0	0.00021092905918	4.10^{-12}

Acknowledgement

The author wishes to express his thanks to the referee for valuable comments, particularly concerning the derivation of the orthogonal polynomials of section 3. The calculations of the numerical example are carried out on the IBM 360/44 of the Computing Centre of the University of Leuven.

REFERENCES

[1] R. E. Bellman, R. E. Kalaba and J. Lockett, *Numerical Inversion of the Laplace Transform*, American Elsevier Publishing Company, New York (1966).
[2] C. Lanczos, *Applied Analysis*, Prentice Hall, Englewood Cliffs, N.J. (1956).
[3] A. Papoulis, A New Method of Inversion of the Laplace Transform. *Quart. Appl. Math.*, 14 (1956) 405–414.
[4] W. T. Weeks, Numerical Inversion of Laplace Transforms using Laguerre Functions, *Journal of the A.C.M.* 13 (1966) 419–426.
[5] H. Dubner and J. Abate, Numerical Inversion of Laplace Transforms and the Finite Cosine Transform, *Journal of the A.C.M.* 15 (1968) 115–123.
[6] H. E. Salzer, Orthogonal Polynomials arising in the Numerical Evaluation of Inverse Laplace Transforms, *M.T.A.C.* 9, (1955) 164–177.
[7] H. E. Salzer, Additional Formulas and Tables for Orthogonal Polynomials originating from Inversion Integrals, *J. Math. Phys.*, 40 (1961) 72–86.
[8] V. I. Krylov and N. S. Skoblya, On the numerical Inversion of the Laplace Transform, *Inzh.-Fiz. Zh.* 4 (1961) 85–101 (Russian).
[9] N. S. Skoblya, Tables for the Numerical Inversion of the Laplace Transform, *Minsk. Izdat. Akad. Nauk BSSSR* (1964) (Russian).
[10] P. J. Davis and P. Rabinowitz, *Numerical Integration*, Blaisdell Publishing Co., Waltham (1967).
[11] A. Erdélyi, *et al.*, *Higher Transcendental Functions*, McGraw-Hill Book Company, New York (1953).
[12] H. L. Krall and O. Frink, A new Class of orthogonal Polynomials: the Bessel Polynomials, *Trans. Amer. Math. Soc.*, 65 (1949) 100–115.
[13] E. Grosswald, On some algebraic properties of the Bessel polynomials, *Trans. Amer. Math. Soc.*, (1951) 197–210.
[14] A. Van Rossum, A note on the location of the zeros of generalized Bessel polynomials and totally positive polynomials. *Nieuw Archief voor Wiskunde* 17 (1969) 142–149.
[15] R. Piessens, *Numerieke metodes voor de inversie van de Laplace Transformatie*, Doctoraatstesis, Katholieke Universiteit Leuven, 1970.
[16] R. Piessens, New Quadrature Formulas for the Numerical Inversion of the Laplace Transform, *BIT*, 9 (1969) 351–361.

[17] A. Stroud and D. Secrest, *Gaussian Quadrature Formulas*, Prentice-Hall, Englewood Cliffs, N.J. (1966).
[18] R. Piessens, Gaussian Quadrature formulas for the Numerical Integration of Bromwich's Integral and the Inversion of the Laplace Transform, *Inst. of Applied Math.*, University of Leuven, (1969).
[19] V. I. Krylov and N. S. Skoblya, *Handbook of Numerical Inversion of Laplace Transforms.* Israel Program for Scientific Translations, Jerusalem (1969) (translated from Russian).

Journal of Engineering Math., Vol. 5 (1971) 1–9